(1) Lecture 2.

I Free groups and free products.

IA Ping-pong lemma. (F. Klein ~1870)

The main tool for finding free subgroups is the following table-tennis lemma:

Ping-pong Lemma. Let G be a group acting on a set X, $\Gamma_1, \Gamma_2 \subset G$ subgroups and Γ be the subgroup of G generated by Γ_1 and Γ_2. Assume that Γ_1 has at least 3 elements and Γ_2 at least 2 elements.

Suppose that there exist two non-empty subsets $X_1, X_2 \subset X$ such that

$$X_2 \not\subset X_1$$

$$\gamma(X_2) \subset X_1 \quad \text{for all } \gamma \in \Gamma_1, \gamma \neq 1$$

$$\alpha(X_1) \subset X_2 \quad \text{for all } \alpha \in \Gamma_2, \alpha \neq 1.$$

Then Γ is isomorphic to the free product $\Gamma_1 * \Gamma_2$.

Proof. Let w be reduced word in alternate form; we have several cases:

(1) $\quad w = a_1 b_1 \cdots a_k, \quad a_i \in \Gamma_1 \setminus \{1\}, \; b_i \in \Gamma_2 \setminus \{1\}.$

then

$$w(X_2) = a_1 b_1 \cdots b_{k-1} a_k (X_2) \subset a_1 b_1 \cdots a_{k-1} b_{k-1}(X_1) \subset \cdots \subset a_1(X_2) \subset X_1$$

since $X_2 \not\subset X_1 \Rightarrow w \neq 1$.

(2) $\quad w = b_1 a_2 b_2 a_2 \cdots b_k$; a_i, b_i as above

Choose $a \in \Gamma_1 \setminus \{1\}$. Then using (1) $\Rightarrow awa^{-1} \neq 1 \Rightarrow w \neq 1$.

(3) $\quad w = a_1 b_1 \cdots a_k b_k$, a_i, b_i as above.

Choose $a \in \Gamma_1 \setminus \{1, a_1^{-1}\}$. using (1) $\Rightarrow awa^{-1} \neq 1 \Rightarrow w \neq 1$.

(4) $\quad w = b_1 a_2 \cdots a_k$, a_i, b_i as above.

choose $a \in \Gamma_1 \setminus \{1, a_k\}$. Using (1) $\Rightarrow awa^{-1} \neq 1 \Rightarrow w \neq 1$. $\square$.

I B Free Semi-groups.

Sometimes one might simply look for large semi-groups inside a given group e.g. when we want to compute the growth of the number of elements. For that reason we can extend ping-pong lemma to this context:

Proposition 1 Let Γ be a group acting on a set X and suppose $\gamma_1, \gamma_2 \in \Gamma$ for which there exist subsets $X_1, X_2 \subset X$ such that:

$$X_1 \cap X_2 = \phi$$
$$\gamma_1 (X_1 \cup X_2) \subset X_1$$
$$\gamma_2 (X_1 \cup X_2) \subset X_2.$$

Then the semi-group of Γ generated by γ_1 and γ_2 is free.

__Proof__ We have to prove that if w, w' are positive words in γ_1, γ_2 then $w = w'$ in Γ implies $w = w'$ as words (in free semi-group on two generators).

- If w is empty word then $w(X_2) = X_2$, $w(X_1) = X_1$. If w' begins with γ_1 $\Rightarrow$ $w'(X_2) \subset X_1$.
If w' begins with γ_2 then $w'(X_1) \subset X_2$. Thus w' is also empty.
- We proceed then by induction on the number of letters in w, w'. We can assume none of w, w' is empty, and thus

$$w = s v \quad , \quad w' = s' v' \quad , \quad s, s' \in \{\gamma_1, \gamma_2\}.$$

Assume $s \neq s'$, for instance $s = \gamma_1$, $s' = \gamma_2$. Then

(1) $w(X_1) = \gamma_1 v(X_1) \subset \gamma_1 v(X_1 \cup X_2) \subset \gamma_1 \cancel{\gamma}(X_1 \cup X_2) \subset X_1$

because for any element v we have $v(X_1 \cup X_2) \subset X_1 \cup X_2$ since

$$\gamma_1 (X_1 \cup X_2) \subset X_1 \cup X_2 \text{ and } \gamma_2 (X_1 \cup X_2) \subset X_1 \cup X_2.$$

(2) $w'(X_1) \subset \gamma_2 (X_1 \cup X_2) \subset X_2$.

Therefore $w \neq w'$ in Γ.

This proves that $s = s'$ and now $w \underset{\Gamma}{=} w'$ implies $v \underset{\Gamma}{=} v'$ by left multiplication (in Γ) by s^{-1}. The induction hypothesis settles the claim. $\square$

__I C__ Sanov's theorem on free subgroups of $SL(2, \mathbb{Z})$.

Theorem (Sanov, 1947) The matrices $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ generate a subgroup of $SL(2, \mathbb{Z})$ which is free of rank 2.

__Proof__. Consider the subgroups generated by each of the two elements, namely

$$\Gamma_1 = \left\{ \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}, n \in \mathbb{Z} \right\} \subset SL(2, \mathbb{Z}), \quad \Gamma_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 2n & 1 \end{pmatrix}, n \in \mathbb{Z} \right\} \subset SL(2, \mathbb{Z}).$$

(3) They are obviously cyclic infinite groups.

We want to use ping-pong lemma. Take for $X = \mathbb{R}^2$ with the usual $SL(2,\mathbb{Z})$ action on $\mathbb{R}^2$. Set then

$$X_1 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 ; \quad |x| > |y| \right\}$$

$$X_2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 ; \quad |x| < |y| \right\}.$$

so that $X_1 \cap X_2 = \phi$. Then let $\begin{pmatrix} x \\ y \end{pmatrix} \in X_2 \Rightarrow$ take any $n \neq 0 \Rightarrow$

$$\begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2ny \\ y \end{pmatrix} \in X_1 \text{ indeed}$$

Since $|x| < |y| \Rightarrow |x+2ny| \geq 2n|y| - |x| > |y|$.

Thus $\gamma X_2 \subset X_1 \quad \forall \gamma \in \Gamma_1 \setminus \{1\}$, and similar proof $\gamma X_1 \subset X_2 \ \forall \gamma \in \Gamma_2 \setminus \{1\}$

$\square$

Remark: The result generalizes easily to ~~pr~~ the fact that the subgroup generated by $\left\{ \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \right\}$ of $SL(2,\mathbb{Z})$, for $\underline{k \geq 2}$ is free.

However this is not true for $k=1$ because the subgroup in question is just $SL(2,\mathbb{Z})$.

Exercise 1) Prove that the subgroup generated by $\left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \right\}$ of $SL(2,\mathbb{C})$ is free provided that $z \in \mathbb{C}$ has

$$|z| \geq 2.$$

2) Find other values of z for which the group is free (e.g. z transcendental etc).

ID

Proposition The Baumslag-Solitar group $BS(1,2)$ has a free semi-group embedded.

Proof $BS(1,2)$ was obtained as the subgroup of $GL(2)$ generated by $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ i.e.

$$BS(1,2) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \ a = 2^n, b \in \mathbb{Z}[\tfrac{1}{2}], n \in \mathbb{Z} \right\}.$$

There is a natural action of $BS(1,2)$ on $\mathbb{R}$ given by

$\begin{pmatrix} a & b \end{pmatrix} \ \ldots \ ax + b$

If $a \neq 1$ then the element $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ has a unique fixed point $\frac{b}{1-a} \in \mathbb{R}$.
Choose then

$$s_j = \begin{pmatrix} a_j & b_j \\ 0 & 1 \end{pmatrix} \quad \text{with} \quad 0 < a_i < 1, \quad \frac{b_1}{1-a_1} \neq \frac{b_2}{1-a_2}; \quad j \in \{1,2\}.$$

and then open intervals $I_1 \ni \frac{b_1}{1-a_1}$, $I_2 \ni \frac{b_2}{1-a_2}$, $I_1 \cap I_2 = \emptyset$,
and then $I \supset I_1 \cup I_2$.
Replacing s_1, s_2 by s_1^N, s_2^N for N large enough we can assume that

$$s_1(I) \subset I_1, \quad s_2(I) \subset I_2.$$

Then the semi-group ping-pong lemma (Prop 1 above) shows that the free semi-group generated by $\langle s_1, s_2 \rangle$ is embedded in $BS(1,2)$.
The same proof works for $BS(1,n)$.

I E Hecke groups

Def We consider Γ_λ the subgroup of $PSL(2,\mathbb{R})$ generated by the elements

$$a_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \text{ where } \lambda = 2\cos\frac{\pi}{q},\ q \geq 3 \text{ an}$$

integer. Then Γ_λ is called a Hecke group level q.

Proposition We have $\Gamma_\lambda \cong \mathbb{Z}/q\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. In particular

$$PSL(2,\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}.$$

Proof. Set $b_\lambda = a_\tau \tau = \begin{pmatrix} -\lambda & 1 \\ -1 & 0 \end{pmatrix}$. Then $\mathrm{Tr}(b_\lambda) = -\lambda$, $\det(b_\lambda) = 1$ and hence its eigenvalues are $-\exp\left(\pm \frac{i\pi}{q}\right)$. In particular b_λ generates a subgroup G of $PSL(2,\mathbb{R})$ which is cyclic of order q.
Further τ generates a subgroup H of $PSL(2,\mathbb{R})$ which is cyclic of order 2.
We want to apply the ping-pong lemma for G and H. The choice of the set X is related to the fact that $PSL(2,\mathbb{R})$ is the isometry group of the hyperbolic plane. More precisely $PSL(2,\mathbb{R})$ acts on

$$\mathcal{H} = \{ z \in \mathbb{C} \ ; \ \mathrm{Im}(z) > 0 \}$$

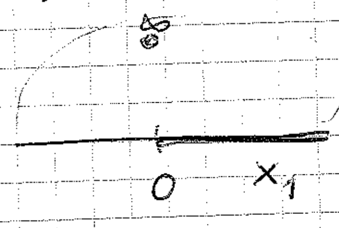
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

Further this action extends to the compactification $\overline{\mathcal{H}} = \mathcal{H} \cup \widehat{\mathbb{R}}$, where

(5) $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is the compactification of the real axis.

Let now $X = \hat{\mathbb{R}}$ so that $\Gamma_\lambda \subset PSL(2,\mathbb{R})$ acts on $\hat{\mathbb{R}}$.

Define next X_1, X_2 with $X_1 \cap X_2 = \phi$:



$$X_1 = (0,\infty) \cup \{\infty\}$$

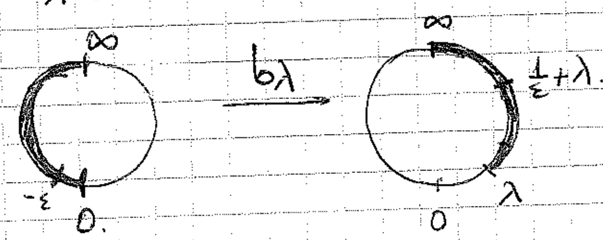
$$X_2 = (-\infty, 0]$$

It is clear that $\sigma(X_1) = X_2$.

We claim that $b_\lambda^k (X_2) \subset X_1$ for $k \leq q-1$.

Observe that $b_\lambda(0) = \infty$, $b_\lambda(\infty) = \lambda$, $b_\lambda(-\varepsilon) = \frac{1}{\varepsilon} + \lambda$ for $\varepsilon > 0$ small.

Thus



$$b_\lambda\big((-\infty, 0]\big) = (\lambda, \infty]$$

Then $b_\lambda^2\big((-\infty,0]\big) = b_\lambda\big((\lambda,\infty]\big) = [\lambda - \frac{1}{\lambda}, \lambda] \subset [0,\lambda]$

and by induction $(*)$.

$$b_\lambda^k\big((-\infty,0]\big) \subset [0,\lambda] \quad \forall\ k \leq q-1.$$

Thus $b_\lambda^k(X_2) \subset X_1$ and Ping-pong lemma proves that

$$\Gamma_\lambda = G * H \simeq \mathbb{Z}/q\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}.$$

Taking $q=3$ then we get $PSL(2,\mathbb{Z})$. $\square$

$(*)$ Induction We prove that

$$b_\lambda^k\big((-\infty,0]\big) = \left[\frac{Q_k}{Q_{k-1}}, \frac{Q_{k-1}}{Q_{k-2}}\right], \quad k \geq 2.$$

where Q_k is the second kind Chebyshev polynomial i.e.

$$Q_k(\cos\theta) = \frac{\sin(k+1)\theta}{\sin\theta}, \text{ where } \theta = \frac{\pi}{q}.$$

$$Q_1(\theta) = \lambda = 2\cos\frac{\pi}{q}. = 2\theta$$

$$Q_0(\theta) = 1$$

$$Q_{k+1}(\theta) = 2\theta Q_k(\theta) - Q_{k-1}(\theta).$$

I F ~~Piecewise linear subgroups~~ Groups of Piecewise linear homeomorphisms of $\mathbb{R}$

Proposition Let f be the PL homeo of $[0,1] \to [0,1]$ given by

$$f(t) = \begin{cases} 4t & \text{if } t \in [0, \frac{1}{5}] \\ \frac{4}{5} + \frac{1}{4}(t - \frac{1}{5}) & \text{if } t \in [\frac{1}{5}, 1]. \end{cases}$$

Let then $\gamma_1 : \mathbb{R} \to \mathbb{R}$ be the homeomorphism

$$\gamma_1(t) = [t] + f(\{t\}) \qquad \forall t \in \mathbb{R}$$

where $[t]$ is the integer part and $\{t\} = t - [t]$ fractionary part of t. Set also

$$\gamma_2 = T \gamma_1 T^{-1}$$

where T is the translation $T(t) = t - \frac{1}{2}$.

Then $\langle \gamma_1, \gamma_2 \rangle$ is a free group in $\mathrm{Homeo}^+(\mathbb{R})$.

__Proof__ Observe first that

$$f^n([\tfrac{1}{5}, 1]) \subset [\tfrac{4}{5}, 1] \quad \text{for } n \geq 1 \quad \text{and}$$

$$f^n([0, \tfrac{4}{5}]) \subset [0, \tfrac{1}{5}] \quad \text{for } n \leq -1.$$

Set $X_1 = \bigcup_{k \in \mathbb{Z}} [k - \frac{1}{5}, k + \frac{1}{5}]$, $\quad X_2 = \bigcup_{k \in \mathbb{Z}} [k + \frac{1}{2} - \frac{1}{5}, k + \frac{1}{2} + \frac{1}{5}] = T(X_1)$

The observations above $\Rightarrow$

$$\gamma_1^n(X_2) \subset X_1 \quad \forall n \in \mathbb{Z} \setminus \{0\}.$$

$$\gamma_2^n(X_1) = T \gamma_1^n T^{-1}(T(X_2)) \subset T(X_1) = X_2 \quad \forall n \in \mathbb{Z} \setminus \{0\}.$$

Ping-pong lemma concludes. $\square$

(I 6) i) Exercises Let $\Gamma \subset PSL(2, \mathbb{R})$ generated by two elements γ_1, γ_2. Then Γ contains a free subgroup if γ_i satisfy one of the following:

(1) γ_1, γ_2 are parabolic with distinct fixed points in $\hat{\mathbb{R}}$.

(2) γ_1 is parabolic with fixed point $x \in \hat{\mathbb{R}}$ and $\gamma_2(x) \neq x$.

(3) γ_1, γ_2 are hyperbolic with fixed points sets disjoint.

ii) An automorphism γ of a tree X is hyperbolic if $\gamma(e) \neq e^{-1}$ $\forall$ edge e and

$$\ell(\gamma) = \min_{x \in X^0} d(x, \gamma x) > 0$$

X^0 set of vertices, d the combinatorial distance in X.

⑦ In this case $\{x\in X^0,\ d(x,\gamma x)=\ell(\gamma)\}\subset X^0$ is the vertex set L^0_γ of a subgraph $L_\gamma\subset X$ which is a geodesic line, called the axis of γ.

If γ_1,γ_2 are hyperbolic automorphisms such that their axes $L_{\gamma_1}, L_{\gamma_2}$ are disjoint then the group of automorphisms generated by γ_1 and γ_2 is free.

__Idea:__ Take the shortest path $y_1 \dots y_2$ joining L_{γ_1} and L_{γ_2}.
Set

$X_1=\{x\in X^0;$ geodesic from x to y_1 contains one of $\gamma_1(y_1),\gamma_1^{-1}(y_1)\}$

$X_2=\{x\in X^0;$ geodesic from x to y_2 contains one of $\gamma_2(y_2),\gamma_2^{-1}(y_2)\}$.

Then we can apply ping-pong lemma to $X_1, X_2, \gamma_1, \gamma_2$.

Remark More generally if $L_{\gamma_1}\cap L_{\gamma_2}$ has finite length ℓ and $m_i\,\ell(\gamma_i)>\ell \Rightarrow \langle\gamma_1^{m_1},\gamma_2^{m_2}\rangle$ is free.

IF Kurosh theorem on subgroups of free products.

We constructed free subgroups into some groups. However sometimes free groups are very ubiquitous. We have in fact a description by:

Kurosh Theorem (1934). Any subgroup $G\subset\Gamma_1*\Gamma_2$ is a free product

$$\mathbb{F}*A_1*A_2*\dots*A_s$$

where $\mathbb{F}$ is a free group, $A_j\subset$ (conjugates in Γ of) Γ_i, $i=1,2$, $j=\overline{1,s}$.

__Ex:__ $G\subset PSL(2,\mathbb{Z})=\mathbb{Z}/3*\mathbb{Z}/2$ implies $G\simeq\mathbb{F}(s)*\underbrace{\mathbb{Z}/3*\dots*\mathbb{Z}/3}_{a}*\underbrace{\mathbb{Z}/2*\dots*\mathbb{Z}/2}_{b}$ for some $s,a,b\in\mathbb{Z}_+$.

Actually all s,a,b can occur.

IH. Let $g=\begin{pmatrix}0&1&0\\1&0&0\\0&0&-1\end{pmatrix}$ $h=\begin{pmatrix}1&0&0\\0&-1/2&\sqrt{3}/2\\0&-\sqrt{3}/2&-1/2\end{pmatrix}$. Then $\langle g,h\rangle\subset SO(3)$ is $\mathbb{Z}/2*\mathbb{Z}/3$ so that $ghgh^2$, gh^2gh generate a free subgroup in $SO(3)$. In fact $\forall k>0$, $n_i\in\{1,2\}$ one proves that

$w=h^{n_1}gh^{n_2}g\dots h^{n_k}g=2^{-k}\begin{pmatrix}p_1&p_2&p_3\sqrt{3}\\q_1&p_4&q_2\sqrt{3}\\q_3\sqrt{3}&p_5\sqrt{3}&q_4\end{pmatrix}$, p_i even integers, q_j odd integers. Any reduced word in g,h besides 1 and g is either of the form w or w^{-1}, wg, gw. $\Rightarrow\langle g,h\rangle\simeq\mathbb{Z}/2*\mathbb{Z}/3$.

(Hausdorff example).

II Tits' alternative

Theorem (Tits 1972) Let $\Gamma \subset GL(n, \mathbb{K})$ finitely generated subgroup ~~finitely generated subgroup~~ for some integer n, $\mathbb{K}$ field of characteristic zero. Then either Γ has a non-abelian free subgroup or Γ has a finite index solvable subgroup.

Rk This is actually true for any subgroup Γ in any Lie group with finitely many components. For instance send the Lie group into GL(Lie algebra) ~~for~~ by the adjoint representation.

II A Tits' alternative in the case of $PSL(2,\mathbb{R})$.

Recall that $PSL(2,\mathbb{R})$ acts as the isometry group of $\mathbb{H}^2$, for instance using ~~the~~ the upper-plane ~~p~~ model $\mathbb{H}^2 = \{z \in \mathbb{C};\ \mathrm{Im}(z) > 0\}$. Then isometries are classified as follows:

- elliptic if they have a fixed point in $\mathbb{H}^2$; then are conjugate to rotations.
- parabolic if it has a unique fixed point in $\widehat{\mathbb{R}}$, conjugate to a translation.
- hyperbolic if it has two fixed points in $\widehat{\mathbb{R}}$; then it has a unique invariant geodesic.

The dynamics of the action on $\widehat{\mathbb{R}}$ is very simple

- if g hyperbolic then there is one attractive point ω and one repulsive point $\Omega \in \widehat{\mathbb{R}}$: $\forall$ neighborhood U of ω and any compact $K \subset \widehat{\mathbb{R}} - \{\Omega\}$ one has $g^k(K) \subset U$ for large enough k, and similarly with g^{-k} for Ω.
- if g parabolic then is one fixed point $p \in \widehat{\mathbb{R}}$: $\forall$ neighborhood $U \ni p$ and K compact in $\widehat{\mathbb{R}} - \{p\}$ we have $g^k(K) \subset U$ for large $|k|$.

Proposition: Let $g, h \in PSL(2,\mathbb{R}) \setminus \{1\}$ be two elements which have no common fixed points in $\mathbb{H} \cup \widehat{\mathbb{R}}$. Then one of the following holds true:

(1) $\langle g, h \rangle \subset PSL(2,\mathbb{R})$ contains a free subgroup.

(2) $\langle g^2 = h^2 = 1 \rangle$ and thus $\langle g, h \rangle$ is the dihedral infinite group.

(3) $g^2 = 1$ and h is hyperbolic s.t. g exchanges the two fixed points of h. Then $\langle g^2 = (gh)^2 = 1 \rangle$ and $\langle g, h \rangle$ is the infinite dihedral group.

Proof Case 1: Assume g is parabolic with fixed point $P \in \widehat{\mathbb{R}}$. Then $k = hgh^{-1}$ is parabolic with fixed point $h(P) \neq P$. Let then

(9)

X_1 (resp X_2) be a compact neighborhood of P (resp $k(P)$) in $\hat{\mathbb{R}}$, $X_1 \cap X_2 = \emptyset$

Then for $n \geq n_0$ the dynamics of a parabolic $\Rightarrow$

$$g^n(X_2) \subset X_1 \quad , \quad k^n(X_1) \subset X_2 \quad , \forall |n| \geq n_0.$$

Ping-pong lemma $\Rightarrow$ $\langle g^n, k^n \rangle$ is free.

Case 2 g, h are hyperbolic. Take X_1, X_2 compact neighborhoods of the fixed points of g and h.

Case 3 h is hyperbolic and g does not exchange its fixed points i.e. $g(Q) = R \notin \{P, Q\}$, where h has fixed pts P, Q.

If $g(P) \notin \{P, Q\}$ as in case 2 for h, ghg^{-1}.

If $g(P) = Q$ consider $g(R)$; if $g(R) \neq P$ case 2 for h, g^2hg^{-2}.

Thus can assume $g(R) = P$. Thus $g' = g^{-1}hg$ is a hyperbolic element with fixed points P and R. Further $g'' = ghg^{-1}hgh^{-1}g^{-1}$ is a hyperbolic element with fixed pts $ghg^{-1}(Q) = Q$ and $ghg^{-1}(P) = S$.

But now $h(R) \neq Q$ and so $S = gh(R) \neq g(Q) = R$; also $h(R) \neq R$, $S \neq g(R) = P$. Thus we apply case 2 to h' and h''.

Case 4 g, h are elliptic with $g^2 \neq 1$. Conjugate g to a rotation around the origin of the disk by some angle $\theta \neq \pi$. Thus $k = hgh^{-1} \neq g$ otherwise h would also fix the origin. Moreover the rotation number $\rho(k)$ is conjugacy invariant hence if $\tilde{k} : \mathbb{R} \to \mathbb{R}$ is a lift of $k : \hat{\mathbb{R}} \to \hat{\mathbb{R}}$, $0 \leq \tilde{k}(0) < 1 \Rightarrow$

$$\rho(k) = \alpha \Rightarrow \min_{x \in \mathbb{R}} \tilde{k}(x) - x \leq \alpha \leq \max_{x \in \mathbb{R}} \tilde{k}(x) - x,$$

and thus there exists $P \in \hat{\mathbb{R}}$ with $k(P) = g(P)$ $\Rightarrow$ $g^{-1}k$ has a fixed point P on $\hat{\mathbb{R}}$ and one of cases 1-3 applies.

Case 5 if $g^2 = h^2 = 1$ then gh generate a cyclic subgroup $\mathbb{Z}$ of index 2 and $\langle g, h \rangle \simeq \langle g, h \mid g^2 = h^2 = 1 \rangle$ which is an extension of $\mathbb{Z}/2$ by $\mathbb{Z}$.

Case 6 if $g^2 = 1$ h hyperbolic, g exchanges fixed points of h $\Rightarrow$

$$ghg^{-1} = h^{-1} \Rightarrow g^2 = (gh)^2 = 1 \Rightarrow \text{case } 5. \ \square$$

II B The rotation number for homeomorphisms of the circle.

Let $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be a homeomorphism. Consider

$\tilde{f}: \mathbb{R} \to \mathbb{R}$ be the lift of f with $0 \le \tilde{f}(0) < 1$.

Definition - Proposition The rotation number $\rho(f)$ is

$$\rho(f) = \lim_{k\to\infty} \frac{1}{k}\left(\tilde{f}^k(x) - x\right)$$

Proof We have $\tilde{f}^j(0) \le \tilde{f}^j(x) \le \tilde{f}^j(0)+1, \quad \forall x \in [0,1]$

and thus

$$(*) \quad \left|(\tilde{f}^j(x)-x) - (\tilde{f}^j(y)-y)\right| \le 1 \qquad \forall x,y \in [0,1]$$

Since $\tilde{f}^j(x) - x$ is periodic of period 1 $(*)$ holds for any $x, y \in \mathbb{R}$.

Let $M = \max|\tilde{f}(x) - x|$; then by induction

$$|\tilde{f}^j(x) - x| \le jM$$

For $k = Lj + r$, $0 \le r < j$, $L > j$ we have then

$$\left|\frac{1}{k}(\tilde{f}^k(x) - x) - \frac{1}{j}\tilde{f}^j(0)\right| < \frac{\text{constant}}{j}$$

$\Rightarrow \lim\limits_{j\to\infty} \frac{1}{j}\tilde{f}^j(0)$ exists and equal to $\lim\limits_{k} \frac{1}{k}(\tilde{f}^k(x) - x) = \rho(f)$. $\square$

Properties of $\rho(f)$:

(a) $\rho(f^h) = h\rho(f)$

(b) $\rho(hfh^{-1}) = \rho(f)$

(c) $\min\limits_x (\tilde{f}(x) - x) \le \rho(f) \le \max\limits_x (\tilde{f}(x) - x)$

(d) $\forall k \; \exists x$ s.t. $\tilde{f}^k(x) - x = k\rho(f)$.

(e) $\rho(f) = 0$ iff f has a fixed point.

II C Corollary $\Gamma \subset PSL(2,\mathbb{R})$ is either solvable or contains a free non-abelian subgroup.

Proof 1) Two elements of $PSL(2,\mathbb{R})$ having a common fixed point in $\mathbb{H}^2 \cup \hat{\mathbb{R}}$ generate a solvable group. This is clear if one element is elliptic since then the other one is elliptic too and they form a subgroup of $SO(2) \cong S^1$. It is clear also if the two elements are parabolic since they generate an abelian group.

Assume one element is parabolic and the other one is hyperbolic. Choose the fixed point ∞. Then the elements are conjugate to $z \to z + k$ (parabolic)

(11)

and $z \to \lambda^2 z$ (the hyperbolic).

Thus the group is contained into $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ which is solvable.

The same argument shows that when both elements are hyperbolic with fixed point ∞ then they are of the form

$$z \to \lambda^2 z + \beta \qquad z \to \mu^2 z + \delta$$

and thus they generate a solvable subgroup of triangular matrices.

2) If Γ does not contain free groups then

- if it contains at least one parabolic isometry $\Rightarrow$ case 1 prop.
- if it contains at least one hyperbolic isometry $\Rightarrow$ all hyperbolics in Γ have a common fixed point (by case 2) and then either Γ infinite dihedral or else all element in Γ have a common fixed point. If Γ contains only elliptic elements $\Rightarrow$ from case 4 that Γ is abelian $\square$.

III Sketch of the proof of Tits' theorem (for free semi-groups instead of groups)

III A. The pb is reduced to subgroups of $GL(n,\mathbb{R})$.

If L is a Lie group with finitely many components the identity component $L_0 \subset L$ is of finite index and so $\Gamma \cap L_0$ has finite index in Γ, so we can suppose L is connected.

The adjoint representation $Ad: L \to GL(g_L)$ where g is the Lie algebra of L is the derivative of the conjugacy action $L \times L \to L$, $h \cdot x = hxh^{-1}$ at the identity element. The exponential map

$$exp : g_L \to L$$

is given by $g \exp(v) g^{-1} = \exp(Ad(g)\, v)$ and its image $\exp(g_L) \subset L$ is an open subset of L. Since the Lie group is an analytic manifold it follows that

$$Ad(g) = 1 \iff [g, \exp(v)] = 1 \;\forall v \in g_L \iff [g, x] = 1 \;\forall x \in L$$

$\iff g$ is central in L. Thus the restriction

$Ad|_\Gamma : \Gamma \to GL(g_L)$ has kernel contained in the center of Γ.

In particular $Ad(\Gamma) \subset GL(\mathfrak{g}_L)$ is virtually solvable / has a free subgroup (non-abelian always) iff Γ is virtually solvable / has a free subgroup. $\square$

III B Consider $\Gamma \subset SL(n, F)$, F is $\mathbb{R}$ or $\mathbb{C}$. Remark that

$$GL(n, \mathbb{R}) \subset SL(2n, \mathbb{R})$$
$$A \longrightarrow \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}.$$

$SL(n, F)$ is the algebraic group given by equation $\det(g) = 1$.

Definition The Zariski topology on an algebraic set X is the topology whose closed sets are the algebraic subsets i.e. the ones given by polynomial equations. We restrict to algebraic sets X into $\mathbb{P}^n(F)$. The Z-closure of $Y \subset X$ is the smallest Zariski closed subset $\overline{Y}$ with $\overline{Y} \supset Y$, and Y is Zariski dense in X if $\overline{Y} = X$.

Remark If $\Gamma \subset SL(n, F)$ is a group then $\overline{\Gamma} \subset SL(n, F)$ is also a subgroup.

III C Dynamics of $SL(n, F)$ elements and sequences

- Set $\mathbb{P}^{n-1}$ for $\mathbb{P}^{n-1}_F$, $[v] \in P^{n-1}$ for the class of vector $v \in F^n$, $[V] \subset P^{n-1}$ the projection of the vector subspace $V \subset F^n$, $V \neq 0$, and $[g] \in PSL(n, F)$ the class of $g \in SL(n, F)$.
- There is a natural metric on P^{n-1} given by

$$d([v], [w]) = \frac{|v \wedge w|}{|v|\ |w|} = \left(\frac{\sum_{i,j} (v_i w_j - v_j w_i)^2}{\sum v_i^2 \ \sum w_i^2} \right)^{1/2}.$$

This is invariant by $SO(n)$ (if $F = \mathbb{R}$) and $SU(n)$ (if $F = \mathbb{C}$), the maximal compact subgroups of $SL(n, F)$. It is only needed to know that

$$d(x, y) = 0 \Rightarrow x = y.$$

[1] Diagonal matrices. Suppose that we have an unbounded sequence $g_i \in SL(n, F)$ which are all diagonal. Specifically

$$g_i = \begin{pmatrix} \lambda_1(i) & & 0 \\ & \ddots & \\ 0 & & \lambda_n(i) \end{pmatrix}, \text{ with } \lambda_1(i) \geq \dots \geq \lambda_n(i) > 0.$$

Then there exists k with the property that

$$\lim_{i \to \infty} \frac{\lambda_k(i)}{\lambda_{k+1}(i)} = \infty$$

In fact use $\lambda_1(i) \cdots \lambda_n(i) = 1$, $\forall i$.

(13) Fix the sequence g_i and thus the k above.

Set also $V_+ = \text{span}(\{e_1, \dots, e_k\}) \subset F^n$, $V_- = \text{span}(\{e_{k+1}, \dots, e_n\}) \subset F^n$

Lemma 0 Let $d_i : \mathbb{P}^{n-1} - [V_-] \longrightarrow \mathbb{R}_+$

$$d_i(v) = d(g_i(v), [V_+]) := \min_{z \in [V_+]} d(g_i(v), z)$$

Then d_i converges uniformly to zero on compacts of $\mathbb{P}^{n-1} - [V_-]$.

Proof. Write $v \in F^n$ as $v = x v_- + y v_+$, $v_- \in V_-$, $v_+ \in V_+$, $|v_-| = |v_+| = 1$. Remark that $g_i(v_\pm) \in V_\pm$ because g_i are diagonal matrices.

We show that

$$(*) \qquad \lim_{i \to \infty} d([g_i(v)], [g_i(v_+)]) = 0, \text{ if } v \in \mathbb{P}^{n-1} \setminus [V_-]$$

In fact $|g_i(v_+)| \geq \lambda_k(i)$, $|g_i(v_-)| < n \lambda_{k+1}(i)$. Thus if $v \notin V_-$

$$|g_i(v) \wedge g_i(v_+)| = x |g_i(v_-) \wedge g_i(v_+)| \leq |x| \, |g_i(v_-)| \, |g_i(v_+)|$$

$$(*) \quad d_i(v) \leq d([g_i(v)], [g_i(v_+)]) \leq \frac{|g_i(v) \wedge g_i(v_+)|}{|g_i(v)| \, |g_i(v_+)|} \leq \frac{|x| \, |g_i(v_-)|}{|g_i(v)|} \leq \frac{|x|}{|y|} \frac{n \lambda_{k+1}(i)}{\lambda_k(i)}$$

since $|g_i(v)| \geq |g_i(v_+)| \cdot |y|$

Moreover if $K \subset \mathbb{P}^{n-1} \setminus [V_-]$ is a compact $\Rightarrow \exists c(K)$ such that $\forall [v] \in K \Rightarrow$

$$y(v) \geq c(K) > 0 \text{ ; taking } |v| = 1 \Rightarrow |x(v)| \leq \sqrt{1 - c(K)^2}$$

and thus $(**) \quad \left| \frac{x(v)}{y(v)} \right| \leq \frac{\sqrt{1 - c(K)^2}}{c(K)}$

Then the convergence in $(*)$ is uniform on compacts. $\square$

[2] Another way to state this lemma is that $g_i(v)$ approach $[V_+]$ whenever $v \notin [V_-]$

However V_+ is a k-dim vector space and not a vector. Thus it is practical to extend the action of $g_i \in SL(n, F)$ from F^n to the k-dimensional subspaces of F^n. Set then

$\mathcal{G} = \mathcal{G}(n,k)$ is the set of k-planes in F^n, called the Grassmannian of k-planes

If $V \in \mathcal{G}(n,k)$ let $\{w_1, \dots, w_k\}$ be a basis of V.

Send them $\imath : \mathcal{G}(n,k) \longrightarrow \mathbb{P}(\Lambda^k F^n)$

$$\imath(V) = [w_1 \wedge \dots \wedge w_k]$$

It is known that $\imath$ is an embedding of $\mathcal{G}(n,k)$ into this projective space, which is actually given by algebraic equations (called Plucker equations)

Further any $g \in SL(n,F)$ induces a map $g : \mathcal{G}(n,k) \to \mathcal{G}(n,k)$, by

$$F^n \supset V \longrightarrow g(V) \subset F^n$$

We can reformulate now lemma above in terms of the new map $g : \mathcal{G} \to \mathcal{G}$ by saying that most $g_i(V)$ approach some point of $\mathcal{G}$. Specifically set $\omega = [V_+] \in \mathcal{G}(n,k)$ and $E = \{ V \in \mathcal{G}(n,k) ; \dim(V \cap V_-) \geq 1 \}$.

Lemma 1. ~~For~~ The sequence $g_i(v)$, $v \in \mathcal{G}(n,k) \setminus E$ converges uniformly on compacts to ω.

Proof If $K \subset \mathcal{G}(n,k) \setminus E$ is a compact then $\bigcup_{W \in K} [W]$ is a compact in $\mathbb{P}^{n-1} \setminus [V_-]$ $\square$

Lemma 2: $E \subset \mathcal{G}(n,k)$ is an algebraic subset.

Proof $W \in E \iff w_1 \wedge \dots \wedge w_k \wedge e_{k+1} \wedge \dots \wedge e_n = 0$ which is a homogeneous algebraic equation in the w_i. $\square$

[3] Arbitrary matrices $g_i \in SL(n,F)$. Recall that any matrix $g \in SL(n,F)$ has the singular value decomposition

$$g = U(g) D(g) U'(g), \quad U, U' \in SO(n) \text{ if } F = \mathbb{R}, \ SU(n) \text{ if } F = \mathbb{C}.$$

and $D(g)$ is diagonal $(\lambda_1 \dots \lambda_n)$, $\lambda_1 \geq \dots \geq \lambda_n > 0$.

Since $K = SO(n)$, $SU(n)$ are compact any sequence $g_i \in SL(n,F)$ ~~which~~ diverges iff $D(g_i)$ diverges. By taking a subsequence we can suppose ~~g~~

$$\lim U(g_i) = U \in K, \quad \lim U'_i(g_i) = U' \in K.$$

Lemma 3: There exists ω (depending on the divergent sequence g_i) $\omega \in \mathcal{G}(n,k)$ and an algebraic subvariety E (depending on g_i) $\subset \mathcal{G}(n,k)$ such that $\lim g_i(v) = \omega$ for any $v \in \mathcal{G}(n,k) \setminus E$, uniformly on compacts.

Proof Take $\omega = U \omega(D(g_i))$, $E = U'^{-1}(E(D(g_i)))$ $\square$.

Examples • $SL(2,\mathbb{R})$, $\mathbb{P}^1$ is a circle, $k=1$ and $\mathcal{G}(2,1)$ is also a circle S^1. ω and Ω are points in S^1. Observe that if $g_i = g^i$ then

- for g parabolic $\omega = E$ is the fixed point of g on $\hat{\mathbb{R}}$
- for g hyperbolic ω is the attractive point, E the repulsive point on $\hat{\mathbb{R}}$

In particular if g is not diagonal ω might belong to E.

(15)

III D Proposition Let $\Gamma \subset SL(n,F)$ be an unbounded Zariski dense subgroup. Then Γ contains a free semigroup.

Proof There exists an unbounded sequence g_i and previous sections yield $\omega = \omega((g_i))$, $E = E((g_i))$, $k \leq n$ verifying lemma 3.

Set $S_0 = \{h \in \Gamma;\ h(\omega) = \omega\}$, $S_{\pm} = \{h \in \Gamma,\ \omega \in h^{\pm 1}(E)\}$.

E is a proper algebraic subset of $\mathcal{G}(n,k)$ and thus $S_0, S_{\pm}$ are proper algebraic subsets of $SL(n,F)$ and so $S_0 \cup S_+ \cup S_- \subsetneq SL(n,F)$, is a proper algebraic subset. Then Γ is not contained in this subvariety because $\overline{\Gamma} = SL(n,F)$. Thus there exists $h \in \Gamma$ with

$$(*) \qquad \omega \notin \{h(\omega)\} \cup h(E) \cup h^{-1}(E).$$

Let now $\beta_i = h g_i$; then $E(\beta_i) = E$, $\omega(\beta_i) = h(\omega) \notin E(\beta_i)$.

Applying $(*)$ once again there exists $g \in \Gamma$ with

$$g(h(\omega)) \neq h(\omega) \qquad g(h(\omega)) \notin E, \quad g^{-1}(h(\omega)) \notin E$$

But now setting $\gamma_i = g\beta_i g^{-1}$ we have $E(\gamma_i) \subset gE \cup g^{-1}E$, $\omega(\gamma_i) = g h(\omega)$

We obtain therefore a sequence of pairs $\beta_i, \gamma_i \in \Gamma$ such that

$$\omega(\beta_i) \neq \omega(\gamma_i)$$

$$\{\omega(\beta_i), \omega(\gamma_i)\} \cap (E(\beta_i) \cup E(\gamma_i)) = \emptyset.$$

Choose then open neighborhoods $B_1^-, B_1^+, B_2^-, B_2^+$ of $E(\beta_i), \omega(\beta_i), E(\gamma_i), \omega(\gamma_i)$ so that

$$B_1^- \cap (B_1^+ \cup B_2^+) = \emptyset$$
$$B_2^- \cap (B_1^+ \cup B_2^+) = \emptyset$$
$$B_1^+ \cap B_2^+ = \emptyset$$

We have $G \setminus B_1^- \cup B_2^- \cup B_1^+ \cup B_2^+ \neq \emptyset$.

Moreover β_i and γ_i uniformly converge to $\omega(\beta_i), \omega(\gamma_i)$ respectively i.e. for large i

$$\beta_i(\mathcal{G} \setminus B_1^-) \subset B_1^+ \qquad \gamma_i(\mathcal{G} \setminus B_2^-) \subset B_2^+$$

Then we can apply ping-pong lemma (or directly one shows that

$$W(G \setminus B_1^- \cup B_2^- \cup B_1^+ \cup B_2^+) \subset B_1^+ \cup B_2^+ \quad \forall\, w \in \langle \beta_i, \gamma_i \rangle_{\text{semigr}}$$

so that the group Γ contains the free semigroup. $\square$

III E A rough idea about the general case

If $\Gamma \subset SL(n,F)$ is unbounded but not Zariski dense, let $\overline{\Gamma} \subset SL(n,F)$ be its Zariski closure which is an algebraic group. Then $G = \overline{\Gamma}$ has a Levi decomposition $G = S \rtimes H$ where S is solvable Lie group and H is reductive Lie group and Γ ~~splits~~ ~~also~~ fits into the exact sequence

$$1 \to S \cap \Gamma \longrightarrow \Gamma \longrightarrow Q \to 1$$

where $Q \subset H$ is Zariski dense subgroup and $S \cap \Gamma \subset S$ is then solvable.

The structure of reductive groups is similar to that of $SL(n,F)$. There is a Cartan decomposition as the singular value decomposition and associated spaces playing the role of the Grassmannian. One has then:

1 – if $Q \subset H$ is unbounded then one proves by similar method that Q contains a free semi-group.

2 – if Q is finite then Γ is virtually solvable.

3 – if $Q \subset H$ is infinite but bounded and then, if $F = \mathbb{R}$ this means that H is compact Lie group.

III F Free semi-groups in compact Lie groups

We assume Γ finitely generated and reduce to the case when the field F

(8) generated by entries of matrices of generators (in $SL(n,\mathbb{R})$) is an algebraic extension of $\mathbb{Q}$ (possibly changing slightly the embedding $\Gamma \to SL(n,F)$).

Consider the norms $\nu : F \to \mathbb{R}_+$ and for each norm ν let F_ν denote the completion of F wrt ν. The ring of integers

$$O_\nu = \{ x ; \ \nu(x) \leq 1 \}$$ ~~is~~ an open subset of F_ν if the norm ν is non-archimedian.

Recall that ν is non-archimedian if $\nu(a+b) \leq \max(\nu(a), \nu(b))$.

Examples: For any $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ let $\sigma : F \to \sigma(F) \subset \mathbb{C}$ be the embedding of F in $\mathbb{C}$ associated to σ. The composition $|\cdot| : \mathbb{C} \to \mathbb{R}_+$ with σ defines an archimedian norm on F and all archimedian norms on F appear this way. We have $F_\nu = \mathbb{R}$ or $\mathbb{C}$.

(17) Further take $F = \mathbb{Q}$ and p a prime. Define

$$\nu_p(x) = p^n \quad \text{where} \quad x = \frac{a}{b \cdot p^n}, \quad \gcd(a,p) = \gcd(b,p) = 1.$$

Then ν_p is non-archimedian and $\mathbb{Q}_p$ is the field of p-adic numbers.

Definition 1) $Nr(F) = \{\nu$ norm on F s.t. $\nu|_{\mathbb{Q}}$ is standard or p-adic$\}$.

2) The ring of Adeles (Weil 1936) is the restricted product

$$\mathbb{A}(F) := \prod_{\nu \in Nr F}^{res} F_\nu = \{(x_\nu) \in \prod_{\nu \in Nr F} F_\nu \ ; \ x_\nu \in O_\nu \text{ for all but finitely many } \nu\}$$

with product topology.

Ex: $F = \mathbb{Q} \Rightarrow \mathbb{A}(\mathbb{Q}) = \mathbb{R} \times \prod_{p \text{ prime}} \mathbb{Q}_p$

since $\forall q \in \mathbb{Q}$ $\exists$ finitely many p s.t. $\nu_p(q) > 1$.

$\boxed{\text{Theorem}}$: The diagonal embedding $F \hookrightarrow \mathbb{A}(F)$ is a discrete subset.

Proof: 0 is isolated point; take $\nu_1, \dots, \nu_m$ the archimedian norms ($\exists$ only finitely many) and circles

$$U = \prod_{i=1}^{m} \{x \in F_\nu \ ; \ \nu_i(x) < \tfrac{1}{2}\} \times \prod_{\mu \in Nr F \setminus \{\nu_1 \dots \nu_m\}} O_\mu$$

which is an open subset. Then for each

$$(x_\nu) \in U \Rightarrow \prod_{\nu \in Nr F} \nu(x_\nu) < \tfrac{1}{2} < 1 \qquad (*)$$

Recall that the product formula states that $\forall x \in F \setminus \{0\}$

$$\prod_{\nu \in Nr F} \nu(x)^{N_\nu} = 1 \quad \text{where } N_\nu = [F_\nu : \mathbb{Q}_\nu]$$

Thus $(*)$ implies that $x = 0$. Thus $U \cap F = \{0\}$. $\square$.

Corollary: $\Gamma \hookrightarrow G(F) \hookrightarrow G(\mathbb{A}(F))$ is discrete subgroup.
G Lie group

Thus if $\forall \nu$, $p_\nu(\Gamma) \subset G(F_\nu)$ is relatively compact $\Rightarrow \Gamma$ is a discrete compact subset of $G(\mathbb{A}(F)) \Rightarrow \Gamma$ finite. Thus $\exists \nu$ s.t. $p_\nu(\Gamma)$ is unbounded (not relatively compact) and can argue as above...